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SATELLITE-TO-SATELLITE TRACKING IN THE HIGH-LOW MODE: LINE-OF-SIGHT ACCELERATION IN A RESIDUAL GRAVITY FIELD

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this study, the residual line-of-sight acceleration for a general satellite configuration is developed rigorous to within the first-order differentials. In addition to the usual "basic term" treated in geophysical literature, two kinds of corrective terms are derived: the term c_1 (related to satellite velocities) and the combined term $c_2+c_3+c_4$ (related to satellite positions). Subsequently, the general formulation is specialized for a high-low configuration. The outcome of computer simulations performed for this

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configuration confirms the first-order formulation. In all cases examined (two arcs of up to three minutes in duration, three different reference gravity fields), the first-order results agree with the known "true" values to within 0.00001 mgal. These results clearly indicate that the basic term lacks the accuracy to represent alone a valid mathematical model. However, when this term is corrected by c_1, its accuracy improves significantly, attaining the level of 0.01 mgal or better. The practical significance of this finding, as well as a possibility to exploit it efficiently in the future, are indicated.

TABLE OF CONTENTS

1. INTRODUCTION

An important quantity obtainable from satellite-to-satellite tracking (SST) is the line-of-sight acceleration. The utilization of this quantity has been attractive to geophysicists especially because the line-of-sight acceleration can be expressed almost entirely in terms of the gradient of the disturbing potential residual to a chosen reference gravity field. In previous reports and papers, approximations were introduced into the line-of-sight acceleration model which transformed the above qualifier "almost entirely" into "entirely". If warranted, such approximations are of great practical value since they lead to an efficient adjustment.

Having the importance of a rigorous line-of-sight acceleration model in mind, we address the issue of errors in the approximate model from both the analytical and the computational standpoints, the latter via results of computer simulations. The emphasis in this task is on SST in the high-low mode, a concept that has grown popular among geophysicists as is documented, e.g., by Jekeli and Upadhyay [1990]. These authors, as well as Gleason [1991], envision the "low" satellite as the space shuttle (or a satellite at the shuttle's altitude), and the "high" satellite as one (or more) of the GPS satellites. Due to the great altitude of the "high" satellite, the gradient of the disturbing potential residual to a sufficiently detailed reference gravity field, such as that represented by a (6,6) or an (8,8) spherical-harmonic expansion, is negligibly small at points along the "high" orbit. This allows for a simplification whereby the gradient of the disturbing potential may be considered only at observation points along the "low" orbit. However, the derivations in the present study will proceed without this simplification, which will be introduced only as a last step.

The ensuing "last-step" formula, developed later on as equation (31), is written in the form

$$\delta\ddot{\rho} = -\delta\underline{\ddot{X}}_1 \cdot \underline{e}^C + \ldots .$$

Here and throughout, the notation δ identifies a given quantity in a residual gravity field, i.e., the difference between this quantity in the actual, or complete, gravity field (where its symbol has no superscript) and the same

quantity as computed in a reference gravity field (where its symbol has the superscript c). Thus, we have

$$\delta\ddot{\rho} = \ddot{\rho} - \ddot{\rho}^c = \text{residual line-of-sight acceleration},$$

where $\ddot{\rho}$ is the line-of-sight acceleration in the complete gravity field and $\ddot{\rho}^c$ is its counterpart in the reference gravity field; and

$$\underline{\delta\ddot{X}}_1 = \underline{\ddot{X}}_1 - \underline{\ddot{X}}_1^c ,$$

where, in the inertial coordinate system (X,Y,Z), $\underline{\ddot{X}}_1 \equiv (\ddot{X}_1, \ddot{Y}_1, \ddot{Z}_1)$ is the acceleration vector of the first (low) satellite in the complete gravity field and $\underline{\ddot{X}}_1^c$ is its counterpart in the reference gravity field; in addition, $\underline{e}^c$ is the line-of-sight unit vector as computed in the reference gravity field.

In following the derivation presented in [Hajela, 1978], Gleason [1991] introduces the approximations alluded to earlier, and obtains the result corresponding to the above "last-step" formula with the dots removed (cf. equation 3, ibid.):

$$\delta\ddot{\rho} \approx -\underline{\delta\ddot{X}}_1 \cdot \underline{e}^c .$$

This outcome is reached also by Jekeli and Upadhyay [1990], whose derivation agrees with that in [Rummel, 1980]. Thus, all four references just mentioned arrive at the same approximate model. Gleason [1991] uses it in the intermediate form (implied by equations 3 and 10, ibid.):

$$\delta\ddot{\rho} = -\underline{\nabla T}(X,Y,Z) \cdot \underline{e}^c ,$$

where $\underline{\nabla T}(X,Y,Z)$ is the gradient, expressed in inertial coordinates, of the disturbing potential at the location of the "low" satellite. From the known time t associated with $\delta\ddot{\rho}$, he then expresses $\underline{\nabla T}$ in earth-fixed polar coordinates. Upon computing a priori covariances as needed for the collocation adjustment model, he uses the latter to obtain an optimal solution for a set of $\bar{\delta}_r$, the mean radial components of the surface gravity disturbances.

Clearly, the accuracy of the results from simulated observations as in [Gleason, 1991], and, especially, the accuracy of the gravity-field determination where the observables are actual line-of-sight accelerations hinges on the accuracy of the mathematical model. In order to design a rigorous procedure for adjusting such accelerations, one has to express the form of, and

evaluate the effect of, the terms represented by the dots in the "last-step" formula presented at the outset. Some, perhaps most, of these terms may be negligible in general. Others may be negligible upon certain restrictions (on the time interval between the epoch of observation and the epoch at which the state vector is given, on the minimal degree and order of the reference gravity field, etc.). However, if there exists a term that cannot be made negligibly small, preferably much smaller than 0.1 mgal characterizing the magnitude of the noise in an electronically-induced signal, this term should be a part of the mathematical model. In theory, it should then be taken into account in the formation of observation equations, a priori covariances, etc.

Some of the results presented by Gleason [1991] indicate that not all of the terms represented by the dots (discussed above) are negligible. When he attributed zero errors to observables simulated in one-second intervals within a 10°-square equatorial oceanic area, he recovered the "ground truth" in 1° squares with no better than a 1.1 mgal r.m.s. error and a 3.8 mgal maximum error (worse results were obtained for lower observational density). In performing comparable simulations over a larger (15°×20°) and geophysically less tranquil continental area, he arrived instead at a 1.6 mgal r.m.s. error and a 6.0 mgal maximum error. It is likely that the cited errors are imputable to the above modeling approximations, especially upon considering the magnification of errors caused by the downward-continuation problem. Therein lies the motivation for a refinement of the traditional line-of-sight acceleration model.

The present study concentrates on the development of the mathematical model rigorous to within the first-order differential quantities. Two second-order differential quantities will appear (temporarily) incidental to the derivations, while further second- and higher-order differential quantities will be mentioned without being expressed explicitly. Although the "last-step" formula implies the simplification whereby the position vector, the velocity vector, and the acceleration vector of the "high" satellite in the complete gravity field are equal to their counterparts computed in the reference gravity field, this simplification will be used neither in the derivations nor in the simulations. The simulations will encompass two "low" arcs passing over the above-mentioned oceanic area, each limited to three minutes in duration from the epoch of a given state vector. The second arc will be used in conjunction with the reference fields represented by (6,6), (8,8), and (10,10) spherical-harmonic

expansions, whereas for the first arc we will consider only the (6,6) reference field. The simulations will serve not only for the verification of the derived mathematical model, but also for the assessment and possible simplifications of this model in view of a sufficiently rigorous, yet computationally manageable adjustment of the line-of-sight accelerations in the high-low mode.

As a historical note, the first part of this report (Chapters 2 - 5) was presented in a similar form in a GL (AFSC) internal paper, "Satellite-to-Satellite Tracking in the High-Low Mode: Line-of-Sight Acceleration in a Residual Gravity Field", April 1990. Two months later, a GL scientist, Mr. David M. Gleason, supplied the author with computer simulations supporting the theoretical findings. An expanded version of the paper including a summary of the computer simulations was published as the author's Status Report No. 5 under the present contract, period 4 April 1990 - 3 July 1990. A year later, another PL (formerly GL) scientist, Dr. Christopher Jekeli, derived an equivalent algebraic result for $\delta\ddot{\rho}$ by taking a formal differential of $\ddot{\rho}$. However, the form of this result is not considered final. A similar derivation is presented in the Appendix herein, where the differential is transformed with the aid of relations from Chapters 3 and 5 into the final formula of Chapter 5 (the general version before the last-step simplification). The Appendix is instructive in confirming the main outcome of Chapter 5 via a semi-independent route, shorter overall than the route of Chapters 2 - 5. Other points of discussion are presented in the Appendix itself.

2. GEOMETRIC QUANTITIES IN THE COMPLETE GRAVITY FIELD

The notation in this analysis is adopted for the most part from [Rummel, 1980]. Thus, the inertial axes are denoted by the letters X, Y, and Z, as are the satellite coordinates; the latter are attributed the subscript 1 for the first satellite (here low), and the subscript 2 for the second satellite (here high). The first-order time derivative is identified by a dot, and the second-order time derivative, by a double dot. Accordingly, the basic symbolism is presented as follows:

$$\text{Sat. 1:}\quad \underline{X}_1 = (X_1, Y_1, Z_1),\quad \dot{\underline{X}}_1 = (\dot{X}_1, \dot{Y}_1, \dot{Z}_1),\quad \ddot{\underline{X}}_1 = (\ddot{X}_1, \ddot{Y}_1, \ddot{Z}_1);$$

$$\text{Sat. 2:}\quad \underline{X}_2 = (X_2, Y_2, Z_2),\quad \dot{\underline{X}}_2 = (\dot{X}_2, \dot{Y}_2, \dot{Z}_2),\quad \ddot{\underline{X}}_2 = (\ddot{X}_2, \ddot{Y}_2, \ddot{Z}_2),$$

where the underlined quantities are vectors. The vector differences in position, velocity, and acceleration are symbolized by

$$\underline{X}_{12} = \underline{X}_2 - \underline{X}_1\ ,\qquad \dot{\underline{X}}_{12} = \dot{\underline{X}}_2 - \dot{\underline{X}}_1\ ,\qquad \ddot{\underline{X}}_{12} = \ddot{\underline{X}}_2 - \ddot{\underline{X}}_1\ . \tag{1}$$

As will become apparent, the development is identical whether the low-low or the high-low configuration is considered. The latter case, however, allows for a simplification that will result in equations (31)-(32d).

The quantities that may be subject to observation, separately or in combination, are

ρ ... line-of-sight range;

$\dot{\rho} = d\rho/dt$ ... line-of-sight velocity (or range rate); and

$\ddot{\rho} = d\dot{\rho}/dt = d^2\rho/dt^2$ ... line-of-sight acceleration.

All of the above symbolism implies the complete (unabridged) gravity field. The present analysis is concerned with $\ddot{\rho}$, especially in view of expressing the latter in a residual gravity field. The line-of-sight range is given by

$$\rho = (\underline{X}_{12}\cdot\underline{X}_{12})^{1/2} = |\underline{X}_{12}|\ , \tag{2}$$

which is used in forming $\underline{e}$, the unit vector in the direction Sat. 1 - Sat. 2:

$$\underline{e} = \rho^{-1}\,\underline{X}_{12}\ . \tag{3}$$

The notation such as $\underline{X}_{12}^2$ is avoided here; instead, this quantity would be written as $\underline{X}_{12}\cdot\underline{X}_{12}$ or $|\underline{X}_{12}|^2$. A similar statement applies with regard to any other vector.

The time derivative of ρ follows from (2) as

$$\dot{\rho} = \rho^{-1}\, \underline{X}_{12}\cdot\dot{\underline{X}}_{12} \ ,$$

or

$$\underline{\dot{\rho} = \dot{\underline{X}}_{12}\cdot\underline{e}} \ . \tag{4}$$

Equation (3) yields the time-derivative of $\underline{e}$:

$$\dot{\underline{e}} = (d\rho^{-1}/dt)\ \underline{X}_{12} + \rho^{-1}\ \dot{\underline{X}}_{12} = -\rho^{-2}\ \dot{\rho}\ \underline{X}_{12} + \rho^{-1}\ \dot{\underline{X}}_{12} \ .$$

or

$$\dot{\underline{e}} = \rho^{-1}(\dot{\underline{X}}_{12} - \dot{\rho}\ \underline{e}) \ . \tag{5}$$

We notice that $\dot{\underline{e}}\cdot\underline{e}=0$ so that the vector $\dot{\underline{e}}$ is perpendicular to $\underline{e}$. From (4) it follows that

$$\ddot{\rho} = \ddot{\underline{X}}_{12}\cdot\underline{e} + \dot{\underline{X}}_{12}\cdot\dot{\underline{e}} \ , \tag{6a}$$

which, due to (5) and (4), becomes

$$\underline{\ddot{\rho} = \ddot{\underline{X}}_{12}\cdot\underline{e} + \rho^{-1}(|\dot{\underline{X}}_{12}|^2 - \dot{\rho}^2)} \ . \tag{6b}$$

As a matter of interest, (6b) can be given in an alternate form. In writing

$$\dot{\underline{X}}_{12} = (\dot{\underline{X}}_{12}\cdot\underline{e})\underline{e} + \dot{\underline{X}}_{12,n} = \dot{\rho}\ \underline{e} + \dot{\underline{X}}_{12,n} \ ,$$

where $\dot{\underline{X}}_{12,n}$ is the component of $\dot{\underline{X}}_{12}$ perpendicular to $\underline{e}$, we have

$$|\dot{\underline{X}}_{12}|^2 = \dot{\underline{X}}_{12}\cdot\dot{\underline{X}}_{12} = \dot{\rho}^2 + |\dot{\underline{X}}_{12,n}|^2 \ ,$$

and (6b) can be presented as

$$\ddot{\rho} = \ddot{\underline{X}}_{12}\cdot\underline{e} + \rho^{-1}|\dot{\underline{X}}_{12,n}|^2 \ . \tag{6b'}$$

3. GEOMETRIC QUANTITIES IN A RESIDUAL GRAVITY FIELD

We now introduce a reference field and denote the quantities computed in it by a superscript c, such as in $\underline{X}_1^c$, $\dot{\underline{X}}_1^c$, $\ddot{\underline{X}}_1^c$; $\underline{X}_2^c$, $\dot{\underline{X}}_2^c$, $\ddot{\underline{X}}_2^c$; and

$$\underline{X}_{12}^c = \underline{X}_2^c - \underline{X}_1^c , \qquad \dot{\underline{X}}_{12}^c = \dot{\underline{X}}_2^c - \dot{\underline{X}}_1^c , \qquad \ddot{\underline{X}}_{12}^c = \ddot{\underline{X}}_2^c - \ddot{\underline{X}}_1^c . \tag{7}$$

In analogy to (2) and (3), one has

$$\rho^c = (\underline{X}_{12}^c \cdot \underline{X}_{12}^c)^{1/2} , \tag{8}$$

$$\underline{e}^c = (\rho^c)^{-1} \, \underline{X}_{12}^c . \tag{9}$$

Precisely the same procedure that resulted in (4), but carried out in the reference field, yields

$$\dot{\rho}^c = \dot{\underline{X}}_{12}^c \cdot \underline{e}^c . \tag{9'}$$

For the "high" satellite, it is usual to adopt $\underline{X}_2^c = \underline{X}_2$, $\dot{\underline{X}}_2^c = \dot{\underline{X}}_2$, and $\ddot{\underline{X}}_2^c = \ddot{\underline{X}}_2$, but here this simplification will be left for the very end. In differencing (4) and (9'), we obtain

$$\dot{\rho} - \dot{\rho}^c = \dot{\underline{X}}_{12} \cdot \underline{e} - \dot{\underline{X}}_{12}^c \cdot \underline{e}^c . \tag{10}$$

The difference on the left-hand side of (10), which owes its existence to the residual gravity field, is symbolized by $\delta\dot{\rho}$:

$$\delta\dot{\rho} = \dot{\rho} - \dot{\rho}^c . \tag{11}$$

This quantity is called the residual line-of-sight velocity. In order to bring (10) to a convenient form, one makes the following approximation:

$$\underline{e} \simeq \underline{e}^c . \tag{12}$$

Thus, (10) is written as

$$\underline{\delta\dot{\rho} = \delta\dot{\underline{X}}_{12} \cdot \underline{e}^c} , \tag{13}$$

where

$$\delta\dot{\underline{X}}_{12} = \dot{\underline{X}}_{12} - \dot{\underline{X}}_{12}^c . \tag{14}$$

Although the symbol $\simeq$ has been replaced in (13) by $=$, the approximation present in (13) should be kept in mind.

In a similar vein, the procedure that resulted in (6b), but carried out in the reference field, gives

$$\ddot{\rho}^c = \underline{\ddot{X}}^c_{12}\cdot\underline{e}^c + (\rho^c)^{-1}[|\underline{\dot{X}}^c_{12}|^2 - (\dot{\rho}^c)^2] \ . \tag{15}$$

In subtracting this equation from (6b), and introducing an additional approximation akin to (12), namely

$$\rho \simeq \rho^c \ , \tag{15'}$$

we obtain

$$\delta\ddot{\rho} = \underline{\delta\ddot{X}}_{12}\cdot\underline{e}^c + (\rho^c)^{-1}[|\underline{\dot{X}}_{12}|^2 - |\underline{\dot{X}}^c_{12}|^2 - \dot{\rho}^2 + (\dot{\rho}^c)^2] \ , \tag{16}$$

where

$$\delta\ddot{\rho} = \ddot{\rho} - \ddot{\rho}^c \ , \qquad \underline{\delta\ddot{X}}_{12} = \underline{\ddot{X}}_{12} - \underline{\ddot{X}}^c_{12} \ .$$

The quantity $\delta\ddot{\rho}$ is the residual line-of-sight acceleration. Working in terms of $\underline{e}^c$, ρ^c, etc., rather than in terms of $\underline{e}$, ρ, etc., has the practical advantage in that the reference field is known and allows for the actual computation of such quantities. We re-write (16) as

$$\underline{\delta\ddot{\rho} = \underline{\delta\ddot{X}}_{12}\cdot\underline{e}^c + (\rho^c)^{-1}\ \delta T} \ , \tag{16'}$$

where δT represe~~ the quantity inside the brackets of (16):

$$\delta T = |\underline{\dot{X}}_{12}|^2 - |\underline{\dot{X}}^c_{12}|^2 - \dot{\rho}^2 + (\dot{\rho}^c)^2 \ . \tag{16''}$$

At a later stage, the approximations (12) and (15') will be removed in order to obtain results rigorous to the first order, i.e., to within the first differentials. Corrections due to both approximations will affect (16) and (16') directly; subsequently, a correction due to (12) will affect δT via (13).

4. REFORMULATION OF THE RESIDUAL LINE-OF-SIGHT ACCELERATION

We consider the vector $\underline{\delta\dot{X}}_{12}$ from (14), which can be written in several forms:

$$\underline{\delta\dot{X}}_{12} = \underline{\dot{X}}_{12} - \underline{\dot{X}}_{12}^{c} = \underline{\dot{X}}_{2} - \underline{\dot{X}}_{1} - (\underline{\dot{X}}_{2}^{c} - \underline{\dot{X}}_{1}^{c})$$

$$= \underline{\dot{X}}_{2} - \underline{\dot{X}}_{2}^{c} - (\underline{\dot{X}}_{1} - \underline{\dot{X}}_{1}^{c}) = \underline{\delta\dot{X}}_{2} - \underline{\delta\dot{X}}_{1} \; , \qquad (17)$$

where the last two symbols have been introduced as

$$\underline{\delta\dot{X}}_{1} = \underline{\dot{X}}_{1} - \underline{\dot{X}}_{1}^{c} \; , \qquad \underline{\delta\dot{X}}_{2} = \underline{\dot{X}}_{2} - \underline{\dot{X}}_{2}^{c} \; .$$

In returning to the formalism of (14), we have

$$|\underline{\delta\dot{X}}_{12}|^{2} = (\underline{\dot{X}}_{12} - \underline{\dot{X}}_{12}^{c})\cdot(\underline{\dot{X}}_{12} - \underline{\dot{X}}_{12}^{c}) = |\underline{\dot{X}}_{12}|^{2} - 2\underline{\dot{X}}_{12}\cdot\underline{\dot{X}}_{12}^{c} + |\underline{\dot{X}}_{12}^{c}|^{2} \; ,$$

and, accordingly,

$$|\underline{\dot{X}}_{12}|^{2} = |\underline{\delta\dot{X}}_{12}|^{2} + 2\underline{\dot{X}}_{12}\cdot\underline{\dot{X}}_{12}^{c} - |\underline{\dot{X}}_{12}^{c}|^{2} \; .$$

If this expression is substituted into (16"), it follows that

$$\delta T = |\underline{\delta\dot{X}}_{12}|^{2} + 2\underline{\dot{X}}_{12}\cdot\underline{\dot{X}}_{12}^{c} - 2|\underline{\dot{X}}_{12}^{c}|^{2} - \dot{\rho}^{2} + (\dot{\rho}^{c})^{2} \; ,$$

or, upon recalling (14) and the fact that $|\underline{\dot{X}}_{12}^{c}|^{2}=\underline{\dot{X}}_{12}^{c}\cdot\underline{\dot{X}}_{12}^{c}$:

$$\underline{\delta T = |\underline{\delta\dot{X}}_{12}|^{2} + 2\underline{\dot{X}}_{12}^{c}\cdot\underline{\delta\dot{X}}_{12} - \dot{\rho}^{2} + (\dot{\rho}^{c})^{2}} \; . \qquad (18)$$

We first denote the last two terms in (18) by δK, and express them as

$$\delta K = -\dot{\rho}^{2} + (\dot{\rho}^{c})^{2} = -(\dot{\rho} - \dot{\rho}^{c})(\dot{\rho} + \dot{\rho}^{c}) \; .$$

From (11), the first factor on the right-hand side is $\delta\dot{\rho}$ and the second factor is $\delta\dot{\rho}+2\dot{\rho}^{c}$, yielding

$$\delta K = -\delta\dot{\rho}^{2} - 2\dot{\rho}^{c}\,\delta\dot{\rho} \; .$$

Upon adopting $\dot{\rho}^{c}$ from (9') and $\delta\dot{\rho}$ from (13), which contains the approximation alluded to earlier, it follows that

$$\delta K \approx -\delta\dot{\rho}^{2} - 2(\underline{\dot{X}}_{12}^{c}\cdot\underline{e}^{c})(\underline{\delta\dot{X}}_{12}^{c}\cdot\underline{e}^{c}) \; . \qquad (19)$$

Next, we decompose the vector $\delta\dot{\underline{X}}_{12}$ into two constituents, the first along $\underline{e}^C$ and the second normal to it (identified by the subscript n):

$$\delta\dot{\underline{X}}_{12} = (\delta\dot{\underline{X}}_{12}\cdot\underline{e}^C)\underline{e}^C + \delta\dot{\underline{X}}_{12,n} .$$

The dot product of this equation with $\dot{\underline{X}}^C_{12}$ yields

$$\dot{\underline{X}}^C_{12}\cdot\delta\dot{\underline{X}}_{12} = (\dot{\underline{X}}^C_{12}\cdot\underline{e}^C)(\delta\dot{\underline{X}}_{12}\cdot\underline{e}^C) + \dot{\underline{X}}^C_{12}\cdot\delta\dot{\underline{X}}_{12,n} ,$$

of which the first term on the right-hand side is needed in (19). The latter then reads

$$\delta K = -\delta\dot{\rho}^2 - 2\dot{\underline{X}}^C_{12}\cdot\delta\dot{\underline{X}}_{12} + 2\dot{\underline{X}}^C_{12}\cdot\delta\dot{\underline{X}}_{12,n} ,$$

and (18) becomes

$$\underline{\delta T = |\delta\dot{\underline{X}}_{12}|^2 - \delta\dot{\rho}^2 + 2\dot{\underline{X}}^C_{12}\cdot\delta\dot{\underline{X}}_{12,n}} . \tag{20}$$

With this quantity substituted into (16'), the residual line-of-sight acceleration is presented as

$$\underline{\delta\ddot{\rho} = \delta\ddot{\underline{X}}_{12}\cdot\underline{e}^C + (\rho^C)^{-1}(|\delta\dot{\underline{X}}_{12}|^2 - \delta\dot{\rho}^2 + 2\dot{\underline{X}}^C_{12}\cdot\delta\dot{\underline{X}}_{12,n})} . \tag{21}$$

This result is only intermediate, since it does not account for the effect of $\delta\underline{X}_{12} = \underline{X}_{12} - \underline{X}^C_{12}$ considered below. Moreover, it contains two terms (the first two terms inside the last parentheses) corresponding to second-order differentials, which subsequently will be neglected.

5. RESULTS RIGOROUS TO THE FIRST ORDER

To correct two out of three approximations implied in the preceding development, we return to equation (16) and observe that it was obtained from (6b) and (15) upon adding the following expression to (6b):

$$\ddot{\underline{X}}_{12}\cdot\underline{e}^{c} - \ddot{\underline{X}}_{12}\cdot\underline{e} + (\rho^{c})^{-1}(|\dot{\underline{X}}_{12}|^{2} - \dot{\rho}^{2}) - \rho^{-1}(|\dot{\underline{X}}_{12}|^{2} - \dot{\rho}^{2}) \ .$$

Accordingly, the correction (i.e., minus the above expression) is

$$\ddot{\underline{X}}_{12}\cdot\underline{\delta e} + \delta\rho^{-1}(|\dot{\underline{X}}_{12}|^{2} - \dot{\rho}^{2}) \ , \tag{22}$$

where the first term is due to (12), and the second term is due to (15').

Upon consulting (2), one forms

$$\delta\rho = \underline{\delta X}_{12}\cdot\underline{e} \ , \tag{23a}$$

$$\delta\rho^{-1} = -\rho^{-2}\ \underline{\delta X}_{12}\cdot\underline{e} \ . \tag{23b}$$

With the aid of (23b), the differential of (3) is obtained as

$$\underline{\delta e} = \rho^{-1}[\underline{\delta X}_{12} - (\underline{\delta X}_{12}\cdot\underline{e})\underline{e}] \ ,$$

where (3) has also served in a substitution. Since the second term inside the brackets is the vector $\underline{\delta X}_{12}$ projected onto $\underline{e}$, it follows that

$$\underline{\delta e} = \rho^{-1}\ \underline{\delta X}_{12,n} \ , \tag{24}$$

where the subscript n indicates the direction normal to $\underline{e}$. We recall that following (19), n indicated the direction normal to $\underline{e}^{c}$. However, this difference in interpretation is of little consequence at the present stage. In particular, the quantities $\underline{e}$, ρ, etc. in (22)-(24) can be replaced at will by $\underline{e}^{c}$, ρ^{c}, etc., since such approximations introduce errors corresponding only to second- and higher-order differentials. These replacements will henceforth be made without further mention.

Next, with the aid of (4) we develop the last factor in (22):

$$|\dot{\underline{X}}_{12}|^{2} - \dot{\rho}^{2} = \dot{\underline{X}}_{12}\cdot\dot{\underline{X}}_{12} - (\dot{\underline{X}}_{12}\cdot\underline{e})^{2} \ . \tag{25a}$$

Since $\dot{\underline{X}}_{12}$ can be decomposed into two orthogonal constituents in a familiar manner as

$$\dot{\underline{X}}_{12} = (\dot{\underline{X}}_{12}\cdot\underline{e})\underline{e} + \dot{\underline{X}}_{12,n} ,$$

it follows that

$$\dot{\underline{X}}_{12}\cdot\dot{\underline{X}}_{12} = (\dot{\underline{X}}_{12}\cdot\underline{e})^2 + \dot{\underline{X}}_{12}\cdot\dot{\underline{X}}_{12,n} .$$

and (25a) becomes

$$|\dot{\underline{X}}_{12}|^2 - \dot{\rho}^2 = \dot{\underline{X}}_{12}\cdot\dot{\underline{X}}_{12,n} . \tag{25b}$$

The first term in (22), which owes its existence to (12), is denoted c_2, and the second term in (22), which is due to (15'), is denoted c_3. With the aid of (24) and (25b) together with (23b), we write

$$c_2 = (\rho^c)^{-1}\,\ddot{\underline{X}}^c_{12}\cdot\underline{\delta X}_{12,n} , \tag{26a}$$

$$c_3 = -(\rho^c)^{-2}(\underline{\delta X}_{12}\cdot\underline{e}^c)(\dot{\underline{X}}^c_{12}\cdot\dot{\underline{X}}^c_{12,n}) . \tag{26b}$$

An additional correction, denoted c_4, is due to the substitution of $\delta\dot{\rho}$ from (13) into δK in (19), which entails the approximation (12). In particular, in obtaining (19) we have replaced the rigorous value of $\delta\dot{\rho}$ from (10) by the expression in (13), i.e., have deformed $\delta\dot{\rho}$ by

$$\dot{\underline{X}}_{12}\cdot\underline{e}^c - \dot{\underline{X}}_{12}\cdot\underline{e} .$$

Accordingly, the correction for $\delta\dot{\rho}$ will be

$$\dot{\underline{X}}_{12}\cdot\underline{\delta e} .$$

From the relation preceding (19) it is apparent that the correction for δK and thus also for δT will be

$$-2\dot{\rho}^c\,\dot{\underline{X}}_{12}\cdot\underline{\delta e} ,$$

and from (16') it follows that the correction for $\delta\ddot{\rho}$ will be

$$-2(\rho^c)^{-1}\,\dot{\rho}^c\,\dot{\underline{X}}_{12}\cdot\underline{\delta e} .$$

In consulting (24), we finally deduce that

$$c_4 = -2(\rho^c)^{-2}\,\dot{\rho}^c\,\dot{\underline{X}}^c_{12}\cdot\underline{\delta X}_{12,n} . \tag{27}$$

We are now in a position to formulate the final version of $\delta\ddot{\rho}$, rigorous to within the first differentials. In returning to (21), we discard the terms

$$|\delta\underline{\dot{X}}_{12}|^2 - \delta\dot{\rho}^2 ,$$

which are essentially second-order quantities, as was mentioned earlier. (Consistent with this fact, Hajela [1978], Rummel [1980], and Jekeli and Upadhyay [1990] show that the above expression is negligible.) Clearly, all the other second- and higher-order differential quantities generated by the replacements described following equation (24) are neglected as well. Before transcribing (21) with the corrections c_2, c_3, and c_4 included, we present the identity

$$\underline{\dot{X}}^c_{12} \cdot \delta\underline{\dot{X}}_{12,n} = \underline{\dot{X}}^c_{12,n} \cdot \delta\underline{\dot{X}}_{12} , \tag{28}$$

which shows that the index n can be shifted from one constituent to the other. This fact, easy to verify, applies in conjunction with general vectors, not only those shown in (28); here it will be used also for c_2 in (26a) and c_4 in (27). What remains of the second term in (21) will be denoted as c_1, and will be transcribed as indicated in (28). In conjunction with c_3 in (26b), we will use the identity

$$\underline{\dot{X}}^c_{12} \cdot \underline{\dot{X}}^c_{12,n} = \underline{\dot{X}}^c_{12,n} \cdot \underline{\dot{X}}^c_{12,n} = |\underline{\dot{X}}^c_{12,n}|^2 , \tag{28'}$$

which again holds true for general vectors.

In transcribing and completing (21) according to the above, we have

$$\underline{\delta\ddot{\rho} = \delta\underline{\ddot{X}}_{12} \cdot \underline{e}^c + c_1 + c_2 + c_3 + c_4} , \tag{29}$$

where the first-order corrections are

$$\underline{c_1 = 2(\rho^c)^{-1}\, \underline{\dot{X}}^c_{12,n} \cdot \delta\underline{\dot{X}}_{12}} , \tag{30a}$$

$$\underline{c_2 = (\rho^c)^{-1}\, \underline{\ddot{X}}^c_{12,n} \cdot \delta\underline{X}_{12}} , \tag{30b}$$

$$\underline{c_3 = -(\rho^c)^{-2}\, |\underline{\dot{X}}^c_{12,n}|^2\, \underline{e}^c \cdot \delta\underline{X}_{12}} , \tag{30c}$$

$$\underline{c_4 = -2(\rho^c)^{-2}\, \dot{\rho}^c\, \underline{\dot{X}}^c_{12,n} \cdot \delta\underline{X}_{12}} . \tag{30d}$$

Since, in practice, the reference field is sufficient for an accurate description of the position, velocity, and acceleration of the "high" satellite, it is permissible to adopt

$$\underline{\delta X}_{12} = -\underline{\delta X}_1 , \qquad \underline{\delta \dot{X}}_{12} = -\underline{\delta \dot{X}}_1 , \qquad \underline{\delta \ddot{X}}_{12} = -\underline{\delta \ddot{X}}_1 .$$

In this case, the formula (29), (30a-d) simplifies to read

$$\underline{\delta \ddot{\rho} = -\underline{\delta \ddot{X}}_1 \cdot \underline{e}^C + c_1' + c_2' + c_3' + c_4'} , \tag{31}$$

where the first-order corrections are

$$\underline{c_1' = -2(\rho^C)^{-1} \, \underline{\dot{X}}^C_{12,n} \cdot \underline{\delta \dot{X}}_1} , \tag{32a}$$

$$\underline{c_2' = -(\rho^C)^{-1} \, \underline{\ddot{X}}^C_{12,n} \cdot \underline{\delta X}_1} , \tag{32b}$$

$$\underline{c_3' = (\rho^C)^{-2} \, |\underline{\dot{X}}^C_{12,n}|^2 \, \underline{e}^C \cdot \underline{\delta X}_1} , \tag{32c}$$

$$\underline{c_4' = 2(\rho^C)^{-2} \, \dot{\rho}^C \, \underline{\dot{X}}^C_{12,n} \cdot \underline{\delta X}_1} . \tag{32d}$$

6. SIMULATED RESIDUAL LINE-OF-SIGHT ACCELERATIONS, AND ASSESSMENT OF THEIR MODELING

The simulations in this study involve two arcs of the "low" satellite generated by Gleason [1991]. The first arc herein is, in fact, the very first of 143 generated arcs entering the 10°-square equatorial oceanic area bordered by the lines of 5° E, W, N, and S. And the second arc herein is the arc associated with the greatest modeling errors among the 143 arcs; below we refer to it as the "worst" arc. The "low" satellite, denoted Sat. 1, emulates a typical space shuttle orbit of approximately 300 km altitude, while the "high" satellite, denoted Sat. 2, emulates one of the GPS satellites of approximately 20,189 km altitude. Among the GPS satellites, the one chosen for simulations of the residual line-of-sight accelerations is the one which has a minimal zenith distance with respect to the "low" satellite. In [Gleason, 1991] the average of such minimum distances is reported to be 28°. Although the duration of the "low" arcs over the oceanic region of interest is two minutes or less, the two arcs serving in the present study are extended to 181 s. Gleason [1991] uses Rapp's 1981 geopotential coefficients through harmonic degree and order (180,180) to describe a field, which he calls "true", approximating the complete gravity field of the earth, while for the reference gravity field he chooses a (6,6) subset of these coefficients.

The mathematical model for the residual line-of-sight acceleration is (29), (30a-d). The above first arc serves mainly for the determination of the step size used in the Hamming orbit generator, which would yield the time-derivatives with sufficient accuracy for any epoch within three minutes from the epoch of the state-vector. Although in [Gleason, 1991] the step size is one second, here three step sizes have been tested: $\Delta t=1$ s, $\Delta t=0.1$ s, and $\Delta t=0.01$ s. At a desired epoch, the time-derivative is computed by

$$[\text{value}(\text{epoch}+\Delta t) - \text{value}(\text{epoch}-\Delta t)]/2\Delta t \;.$$

The step size $\Delta t=1$ s is deemed too coarse for the precise calculations needed in the model verification. In particular, errors in $\ddot{\delta\rho}$ have been found to range from 0.04 mgal to 0.18 mgal for epochs ranging from 2 s to 179 s from the state-vector epoch. The step size $\Delta t=0.1$ s reduces the magnitude of these

errors by over 90%. To produce results accurate to 0.001 mgal or better, $\Delta t = 0.01$ s is adopted. The step-size errors affect the "true" $\delta\ddot{\rho}$ and the first term on the right-hand side of (29), called "basic term", by approximately the same amount. Thus, the error of the basic-term modeling (i.e., the difference between the basic term and $\delta\ddot{\rho}$), as well as the error of a more complete modeling, are affected by the step-size errors to a much lesser degree than 0.001 mgal, perhaps by two or more orders of magnitude. We conclude that for the modeling-error analysis, the step size $\Delta t = 0.01$ s is entirely satisfactory.

With this step size and a (6,6) reference field, the values of $\delta\ddot{\rho}$ on the first arc have been generated in four intervals from the state-vector epoch: 2 s, 60 s, 120 s, and 179 s. These four epochs will be used throughout the analysis. The simulated values of $\delta\ddot{\rho}$ at these epochs are

$$-2.883\ , \qquad -3.375\ , \qquad -1.697\ , \qquad 3.782\ ,$$

the units being milligals. The corresponding values of the basic term differ from the $\delta\ddot{\rho}$ values by the amounts, called errors, which are listed (in mgal) as

$$0.002\ , \qquad 0.016\ , \qquad -0.043\ , \qquad -0.142\ . \tag{33a}$$

When the term c_1 is included, the errors decrease substantially:

$$\text{-}\cdot\ , \qquad -0.0004\ , \qquad -0.0009\ , \qquad 0.0011\ . \tag{33b}$$

Finally, when also the terms c_2, c_3, and c_4 are included, the errors become

$$\text{--}\ , \qquad -0.000005\ , \qquad 0.000003\ , \qquad 0.000002\ . \tag{33c}$$

The notation "--" indicates completely negligible amounts. Already at this early stage, the validity of the mathematical model (29), (30a-d) is confirmed. The small remaining errors displayed in (33c) stem partly from neglecting the second- and higher-order differentials, and partly from the step-size errors; the round-off errors are negligible here (with the double-precision arithmetic).

At a state-vector epoch the errors are zero, since the position vector $\underline{X}_1$ and the velocity vector $\dot{\underline{X}}_1$ in the complete gravity field are equal to their counterparts in the reference field by construction. (For the "high" satellite this equality holds true, or nearly true, also at epochs different from the state-vector epoch.) Accordingly, at a state-vector epoch we have

$$\underline{X}_{12} = \underline{X}_{12}^{c} , \qquad \dot{\underline{X}}_{12} = \dot{\underline{X}}_{12}^{c} ,$$

which implies that

$$\delta \underline{X}_{12} = 0 , \qquad \delta \dot{\underline{X}}_{12} = 0 ;$$

$$\underline{e} = \underline{e}^{c} , \qquad \rho = \rho^{c} , \; \ldots \; .$$

Thus, all four corrective terms c_1 through c_4 in (29), as well as all second- and higher-order differential terms, are zero, and the basic term supplies the rigorous value at any epoch where the state-vector is given. The main question to be answered is: How far from the state-vector epoch is the basic term accurate enough to represent alone an acceptable mathematical model?

In considering the first arc, and requiring the modeling accuracy to be at least 0.1 mgal, we observe from (33a) the basic terms alone would be acceptable for epochs separated by nearly three minutes from the state-vector epoch. However, the situation depicted by the first arc is overly optimistic. If we consider instead the "worst" arc in the same context, i.e., with a 0.01 s step size and a (6,6) reference field, in lieu of (33a) we obtain

$$0.010 , \qquad 0.269 , \qquad 0.386 , \qquad 0.545 . \tag{34}$$

Since the pertinent oceanic area is geophysically tranquil, the relatively large errors in (34) may be the norm for many other regions. Equation (34) indicates that the basic term alone could represent a (marginally) acceptable model only in the case of observational epochs closer to the state-vector epoch than 30 s.

We now present in detail the results that culminated in (34). This will confirm, once again, the validity of the model (29), (30a-d), and suggest a simplified residual line-of-sight model, although not as simple as the basic term alone. We recall that the step size is 0.01 s, the "true" gravity field is given by (180,180) spherical-harmonic coefficients, the reference field is given by the (6,6) subset of these coefficient, and the results are listed in milligals. Two additional groups of results will be presented, corresponding to (8,8) and (10,10) reference gravity fields. As peripheral information, we list the values of $\delta\ddot{\rho}$ for the three reference fields as obtained for the usual four epochs (2 s, 60 s, 120 s, and 179 s from the state-vector epoch):

(6,6)	2.607 ,	-2.000 ,	0.417 ,	2.038 ;
(8,8)	3.077 ,	-0.423 ,	3.314 ,	6.191 ;
(10,10)	-0.530 ,	-5.069 ,	-1.788 ,	1.130 .

These values do not reveal any clearcut pattern, or significant differences in magnitude, which would single out one reference field in preference to another.

In proceeding in detail with the (6,6) reference gravity field, we will list all the results to six decimals. The term c_1 will be presented separately, while the terms c_2, c_3, and c_4, which all depend on $\underline{\delta X}_{12}$, will be combined into the term $c_2+c_3+c_4$. For the sake of interest, a term containing the second-order differential quantities seen in (21) will also be listed, written as (z):

$$(z) = (\rho^c)^{-1}(|\dot{\underline{\delta X}}_{12}|^2 - \delta\dot{\rho}^2) .$$

Clearly, this term accounts only for some second- and higher-order differential quantities. Its listing will illustrate that it is negligibly small, as already transpired from [Hajela, 1978], [Rummel, 1980], and [Jekeli and Upadhyay, 1990].

The guidance to the four groups of results associated with the "worst" arc is as follows. Above each group is listed the time interval from the state-vector epoch (2 s, 60 s, 120 s, or 179 s), together with the corresponding value of $\delta\ddot{\rho}$ (see also the preceding outcome for the 6,6 reference field, where these values are shown to three decimals). Each group proper begins with the line headed "individual terms", where the first item is the basic term, the second item is the term c_1, the third item is the combined term $c_2+c_3+c_4$, and the fourth item is (z). The next line is headed "modeled $\delta\ddot{\rho}$", where the model is presented in a cumulative fashion; the first item is the repetition of the basic term, the second item is [basic term + c_1], and the third and last item is [basic term + c_1 + $(c_2+c_3+c_4)$], i.e., the complete right-hand side of (29). (The value in the second or the third item is found as the value in the preceding item on the same line plus the value just above the desired item.) Finally, the third line is headed "remaining errors", and it contains the errors in the values representing "modeled $\delta\ddot{\rho}$" of the second line. (Each of the three items on the third line is computed as the value just above it minus the value $\delta\ddot{\rho}$ listed above the first line.) All values are rounded-off to six decimals.

The four groups of results are listed below.

2 s ..... $\delta\ddot{\rho} = 2.607035$

	basic term	c_1	$c_2+c_3+c_4$	(z)	
individual terms:	2.616992	-0.009953	-0.000004	(--)	
modeled $\delta\ddot{\rho}$:	2.616992	2.607039	2.607036		
remaining errors:	0.009957	0.000005	0.000001		(35a)

60 s ..... $\delta\ddot{\rho} = -2.000465$

	basic term	c_1	$c_2+c_3+c_4$	(z)	
individual terms:	-1.731951	-0.265731	-0.002781	(0.000001)	
modeled $\delta\ddot{\rho}$:	-1.731951	-1.997682	-2.000463		
remaining errors:	0.268514	0.002783	0.000002		(35b)

120 s ..... $\delta\ddot{\rho} = 0.416935$

	basic term	c_1	$c_2+c_3+c_4$	(z)	
individual terms:	0.803185	-0.378171	-0.008090	(0.000004)	
modeled $\delta\ddot{\rho}$:	0.803185	0.425013	0.416924		
remaining errors:	0.386250	0.008078	-0.000012		(35c)

179 s ..... $\delta\ddot{\rho} = 2.037552$

	basic term	c_1	$c_2+c_3+c_4$	(z)	
individual terms:	2.582464	-0.530705	-0.014204	(-0.000001)	
modeled $\delta\ddot{\rho}$:	2.582464	2.051759	2.037555		
remaining errors:	0.544912	0.014207	0.000003		(35d)

For the sake of interest, we list the six state-vector components (in inertial coordinates) at the epoch 179 s for the "low" satellite, followed by the six state-vector components at the same epoch for the "high" satellite, both state-vectors having been generated in the "true" field; the position components are given in meters, and the velocity components are given in meters per second.

Sat. 1:	$X = 695{,}616.623$,	$Y = 6{,}628{,}710.831$,	$Z = -190{,}943.856$,
	$\dot{X} = -6{,}772.261608$,	$\dot{Y} = 612.098398$,	$\dot{Z} = -3{,}683.667565$;
Sat. 2:	$X = 10{,}970{,}235.262$,	$Y = 22{,}255{,}878.282$,	$Z = -9{,}466{,}691.250$,
	$\dot{X} = -1{,}651.398895$,	$\dot{Y} = 2{,}029.425766$,	$\dot{Z} = 2{,}857.380367$.

Accordingly, at this epoch we have $|\underline{X}_1| = 6{,}667{,}844$ m, i.e., the altitude of the first satellite is about 290 km; and $|\underline{X}_2| = 26{,}557{,}267$ m, i.e., the altitude of the second satellite is about 20,180 km. For the first satellite, the component differences $\underline{\delta X}_1$ and $\underline{\delta \dot{X}}_1$ are

$$-0.081\ , \quad -0.450\ , \quad -0.743\ ; \quad 0.000370\ , \quad -0.004181\ , \quad -0.007800\ .$$

The first three values reveal that $|\underline{\delta X}_1| = 0.872$ m. For the second satellite the component differences are essentially zero (they are 0.000001 and -0.000001 in X and Y, respectively, smaller otherwise). This is consistent, for example, with the statement that followed equation (30d).

The error values in (35a-d) warrant a discussion. We begin with the third item in each of the lines (35a-d), i.e., with the error remaining in the expression [basic term + c_1 + $(c_2 + c_3 + c_4)$]. This error is exceedingly small for all four epochs, confirming the theoretical formula and indicating that the second- and higher-order differentials are negligible within three minutes from the state-vector epoch (and probably much beyond three minutes), that the step-size errors are inconsequential, and that the orbital simulations are flawless. In fact, the reliability and accuracy of the orbit-generating program perfected by Gleason [1991] is verified here in an independent and interactive fashion (interactive in the sense of high-low satellites, high-low order gravity fields). In continuing with the second item, i.e., with the error remaining in the expression [basic term + c_1], we note that this error is at or below the level of 0.01 mgal for all four epochs. Thus, the corrective term c_1 is sufficient to extend the validity of the basic term to three minutes from the

state-vector epoch, and much beyond three minutes if one is willing to accept modeling errors on the order of 0.1 mgal. Finally, the first item, i.e., the error in the basic term, indicates that the model comprising the basic term alone introduces inadmissible errors already within a few tens of seconds from the state-vector epoch, even if one tolerates the modeling error of 0.1 mgal.

Faced with the unacceptability of the basic term for accurate modeling of the line-of-sight accelerations in the gravity field residual to a (6,6) reference field, one is compelled to ask whether a reference field higher than (6,6) could at least partly remedy this shortcoming. Accordingly, two additional computer runs have been analyzed, utilizing an (8,8) and a (10,10) reference field. First of all, we remark that the errors in the model [basic term + c_1 + (c_2+c_3+c_4)] are again exceedingly small, listed as

(8,8) ..... 0.000005 , 0.000001 , -- , 0.000007 ;

(10,10) .... 0.000006 , 0.000003 , -0.000002, 0.000005 .

There is no essential difference between these errors and the errors presented above for a (6,6) reference gravity field.

Next, we present the errors of the basic term in conjunction with these reference fields, again for the four chosen epochs (2 s, 60 s, 120 s, 179 s). We enhance the usefulness of the results by adding also the errors of the model [basic term + c_1], in parentheses. These two kinds of errors are listed below to three decimals. For the sake of completeness, the errors presented earlier for a (6,6) reference field (see equations 35a-d) are recapitulated using the same format:

(6,6) ..... 0.010 (-) , 0.269 (0.003) , 0.386 (0.008) , 0.545 (0.014) ;

(8,8) ..... 0.005 (--) , 0.108 (0.002) , 0.046 (0.005) , 0.041 (0.010) ;

(10,10) .... 0.008 (-) , 0.188 (0.001) , 0.166 (0.001) , 0.162 ( ) .

Although we notice improvements in the basic term in conjunction with higher order reference fields (as compared to the 6,6 field), the improvements are not monotonous. For example, the error of 0.386 mgal for the (6,6) reference field improves to 0.046 mgal for the (8,8) reference field, but then it grows to 0.166 mgal for the (10,10) reference field.

From the above results it appears that a relatively high-order reference gravity field would have to be utilized to substantially lower the error level of the basic term, if this is indeed possible. As another attempt to make the errors in the basic term acceptable, one could lower the time interval between the state-vector epoch and any observational epoch to, e.g., less than 30 s. (Although in the case of the 8,8 reference field the errors in the basic term are seen to decrease beyond the epoch 60 s, this feature is exceptional as is gleaned from the results for the 10,10 and, especially the 6,6 reference field.) Both these possibilities, whether exploited separately or in combination, would limit the practical usefulness of the mathematical model represented by the basic term alone. Moreover, such a model would still suffer from poor or marginal accuracy.

We conclude the simulation analysis by mentioning that due to computer limitations, one is naturally concerned about the possibility that the simulated errors may be overly optimistic. This possibility stems from the fact that the "true" gravity field has been associated here with a (180,180) field rather than with a (360,360) or a more detailed field. However, we alleviate such concerns by presenting the errors corresponding to a (6,6) reference field (as used earlier) and to a mere (60,60) "true" field:

0.010 (--) , 0.269 (0.003) , 0.380 (0.008) , 0.538 (0.014) .

While $\delta\ddot{\rho}$ at any of the four epochs has changed by 0.2 mgal or less between the (60,60) and (180,180) "true" fields, the errors in the basic term are seen to have changed by 0.007 mgal or less, and the errors in [basic term + c_1] are seen to be unchanged. By far, most of the change in the residual line-of-sight acceleration due to the change in the "true" field is absorbed by the basic term. These considerations, extrapolated to more detailed "true" fields, indicate that the simulated error analysis is likely to be valid with sufficient accuracy also in the actual gravity field of the earth.

7. RECOMMENDATIONS FOR RESIDUAL LINE-OF-SIGHT ACCELERATION MODELING

Guided by the above results and considerations, we envision a manageable, yet accurate mathematical model represented by [basic term + c_1]. We have seen that to within three minutes from the state-vector epoch, and probably much beyond three minutes, the errors in such a model are negligible even for a (6,6) reference gravity field. Two avenues open quite naturally before us. The first is to use the term c_1 as a fully equivalent partner of the basic term in the model, involved in the formation of the design matrix for the collocation adjustment, of a priori covariances, etc.

The second avenue is much simpler and much more efficient. In consulting the detailed results listed in conjunction with the (6,6) reference gravity field, we observe that for the most part the term c_1 is significantly smaller than the basic term. This is all the more so for shorter time intervals from the state-vector epoch, and for higher-order reference fields. Thus, under certain stipulations (which should be subject to further tests, including tests covering additional regions of the globe), the term c_1 could be treated not as a full partner of the basic term, but merely as a small observational correction separate from the model. This conceptual simplification is far-reaching. It makes it possible to revert to the basic term as a highly efficient mathematical model for the residual line-of-sight accelerations by overcoming the initial shortcoming of this model, the low accuracy.

The crucial aspect of this approach is the computation of the correction c_1. In practice, c_1 can be approximated by the same procedure that has been used in this analysis. The true gravity field can be approximated, e.g., by a (180,180) field, and the satellite orbits in question can be approximated by simulated orbits sufficiently close to their true counterparts. When necessary, the approximating orbits can be rectified (to correct for their drift, in time, with respect to the true orbits). The step size can be lowered to 0.1 s, etc. (for better accuracy), but only in the immediate vicinity of the observational epoch; elsewhere the step size can be much coarser in analogy to the technique utilized by Gleason [1991].

Akin to a previous concern about the validity of error analysis using a truncated representation of the gravity field, one may now be concerned about the accuracy of c_1 computed in such a representation. It is assumed that the

spherical-harmonic coefficients providing this representation are reasonably accurate. An example using a (60,60) set of coefficients versus a (180,180) set, where the former is a subset of the latter, again helps to alleviate such concerns. In writing c_1 computed with a (60,60) truncated field in brackets, and in writing c_1 computed with a (180,180) "control" field in braces, we have for the usual four epochs:

$$[-0.010]\{-0.010\}, \quad [-0.266]\{-0.266\}, \quad [-0.372]\{-0.378\}, \quad [-0.524]\{-0.531\}.$$

Accordingly, for an observational epoch separated by three minutes from the state-vector epoch, the effect of the indicated truncation generates a mere 0.007 mgal difference in the value of c_1, corresponding to the outcome seen at the close of the preceding section. We reiterate that by far, most of the change in the residual line-of-sight acceleration is absorbed by the basic term. This behavior of the basic term further supports the concept of using c_1 as an observational correction computed in a reasonably detailed and reasonably accurate "true" gravity field.

In the above setup, an approximation to the true correction c_1 can be obtained according to (30a) or, in practice, according to (32a). In principle, the suggested approach uses an "adjoint model" (operating on assumed satellite orbits in conjunction with an assumed gravity field) alongside the "true model" (operating on the actual satellite orbits governed by the actual gravity field), but only insofar as the correction c_1 is concerned. The advantage of this approach, based on the in-practice ascertainable smallness of c_1 relative to the basic term, is substantial. Due to c_1 being considered as an observational correction, the need for its treatment in view of the design matrix, of a priori covariances, etc., is eliminated.

8. CONCLUSION

In the first part of this study, the residual line-of-sight acceleration for a general satellite configuration is developed rigorous to within the first-order differentials. In addition to the usual "basic term" treated in geophysical literature, two kinds of corrective terms are derived: the term c_1 (related to satellite velocities) and the combined term $c_2+c_3+c_4$ (related to satellite positions). Subsequently, the general formulation is specialized for a high-low configuration. In this case, the term c_1 depends essentially on the residual velocity vector of the "low" satellite at the epoch of observation, and the term $(c_2+c_3+c_4)$ depends essentially on the residual position vector of the same satellite at the same epoch. The above qualifier "residual" implies the difference between a vector expressed in the complete gravity field and its counterpart obtained in the reference gravity field.

In the latter part of the study, the analysis of computer simulations for a high-low configuration confirms the first-order general formula (devoid of approximations involving the "high" satellite). In all cases examined (two arcs of up to three minutes in duration, three different reference gravity fields), the first-order results agree with the known "true" values, obtained with the step size $\Delta t=0.01$ s, to within 0.00001 mgal. The analysis clearly indicates that the basic term lacks the accuracy to represent alone a valid mathematical model; in one realistic simulation its error surpasses 0.5 mgal. However, when this term is corrected by c_1, its accuracy improves significantly, attaining the level of 0.01 mgal or better. Thus, the term $(c_2+c_3+c_4)$ as well as all of the higher-order differentials can safely be neglected in practice. As a further possible simplification, it is recommended that the term c_1 be considered merely as an observational correction in the subsequent (collocation) adjustment, computed in an "adjoint model" where the true gravity field and the true satellite orbits are substituted for by their sufficiently accurate and easily manageable approximations.

APPENDIX

ALGEBRAIC DERIVATION AND A USEFUL TRANSFORMATION OF THE FORMULA GIVING THE RESIDUAL LINE-OF-SIGHT ACCELERATION

To find the formal differential of the line-of-sight acceleration, we first recapitulate the standard relations (2), (3), (4), and (6b):

$$\rho = |\underline{X}_{12}| \ , \qquad \underline{e} = \rho^{-1}\,\underline{X}_{12} \ , \qquad \dot{\rho} = \dot{\underline{X}}_{12}\cdot\underline{e} \ ,$$

$$\ddot{\rho} = \ddot{\underline{X}}_{12}\cdot\underline{e} + \rho^{-1}(|\dot{\underline{X}}_{12}|^2 - \dot{\rho}^2) \ .$$

In forming the differential $\delta\ddot{\rho}$ of $\ddot{\rho}$, use is made of $\delta\rho$, $\delta|\dot{X}_{12}|^2$, and $\delta\dot{\rho}^2$:

$$\delta\rho = \delta(\underline{X}_{12}\cdot\underline{X}_{12})^{1/2} = (1/2)|\underline{X}_{12}|^{-1}\,2\underline{X}_{12}\cdot\underline{\delta X}_{12} = \underline{e}\cdot\underline{\delta X}_{12} \ ,$$

$$\delta|\dot{\underline{X}}_{12}|^2 = \delta(\dot{\underline{X}}_{12}\cdot\dot{\underline{X}}_{12}) = 2\dot{\underline{X}}_{12}\cdot\underline{\delta\dot{X}}_{12} \ ,$$

$$\delta\dot{\rho}^2 = 2\dot{\rho}\,\delta\dot{\rho} = 2\dot{\rho}(\underline{\delta\dot{X}}_{12}\cdot\underline{e} + \dot{\underline{X}}_{12}\cdot\underline{\delta e}) \ .$$

We now readily deduce

$$\begin{aligned}\delta\ddot{\rho} = {} & \underline{\delta\ddot{X}}_{12}\cdot\underline{e} + \ddot{\underline{X}}_{12}\cdot\underline{\delta e} - \rho^{-2}\,\underline{e}\cdot\underline{\delta X}_{12}(|\dot{\underline{X}}_{12}|^2 - \dot{\rho}^2) \\ & + \rho^{-1}[2\dot{\underline{X}}_{12}\cdot\underline{\delta\dot{X}}_{12} - 2\dot{\rho}(\underline{\delta\dot{X}}_{12}\cdot\underline{e} + \dot{\underline{X}}_{12}\cdot\underline{\delta e})] \ ,\end{aligned}$$

where

$$\underline{\delta e} = -\rho^{-2}\,\delta\rho\,\underline{X}_{12} + \rho^{-1}\,\underline{\delta X}_{12} = \rho^{-1}\,\underline{\delta X}_{12} - \rho^{-2}(\underline{e}\cdot\underline{\delta X}_{12})\underline{X}_{12} \ .$$

Since the differential $\delta\ddot{\rho}$ pertains to the reference field identified by the superscript c, all finite quantities above are attributed this superscript:

$$\begin{aligned}\delta\ddot{\rho} = {} & \underline{\delta\ddot{X}}_{12}\cdot\underline{e}^c + \ddot{\underline{X}}^c_{12}\cdot\underline{\delta e} - (\rho^c)^{-2}\,\underline{e}^c\cdot\underline{\delta X}_{12}[|\dot{\underline{X}}^c_{12}|^2 - (\dot{\rho}^c)^2] \\ & + (\rho^c)^{-1}[2\dot{\underline{X}}^c_{12}\cdot\underline{\delta\dot{X}}_{12} - 2\dot{\rho}^c(\underline{e}^c\cdot\underline{\delta\dot{X}}_{12} + \dot{\underline{X}}^c_{12}\cdot\underline{\delta e})] \ ,\end{aligned} \qquad \text{(A.1)}$$

where

$$\underline{\delta e} = (\rho^c)^{-1}\,\underline{\delta X}_{12} - (\rho^c)^{-2}(\underline{e}^c\cdot\underline{\delta X}_{12})\underline{X}^c_{12} \ . \qquad \text{(A.2)}$$

The formulation (A.1,2) is not very insightful or easily tractable. Moreover, we wish to present $\delta\ddot{\rho}$ in a form where the "basic term" $\underline{\delta\ddot{X}}_{12}\cdot\underline{e}^c$ is followed by

the most important of the remaining terms. The analysis of Chapter 6 indicates that such a term contains $\underline{\delta\dot{X}}_{12}$.

In the general formula (29), (30a-d), this term is presented concisely as c_1. Thus, we attempt to transform (A.1,2) into a form such as (29), (30a-d). First, we use equation (9'); the decomposition of a vector into two orthogonal constituents as in the step leading to equation (24); and the identity (28):

$$\underline{\dot{X}}^c_{12}\cdot\underline{\delta\dot{X}}_{12} - \dot{\rho}^c\,\underline{e}^c\cdot\underline{\delta\dot{X}}_{12} = \underline{\dot{X}}^c_{12}\cdot[\underline{\delta\dot{X}}_{12} - (\underline{\delta\dot{X}}_{12}\cdot\underline{e}^c)\underline{e}^c]$$

$$= \underline{\dot{X}}^c_{12}\cdot\underline{\delta\dot{X}}_{12,n} = \underline{\dot{X}}^c_{12,n}\cdot\underline{\delta\dot{X}}_{12}\ .$$

As a result, we can already extract the term c_1 from (A.1):

$$\underline{c_1 = 2(\rho^c)^{-1}\,\underline{\dot{X}}^c_{12,n}\cdot\underline{\delta\dot{X}}_{12}}\ . \qquad (A.3)$$

After this extraction, only the third term remains inside the second pair of brackets in (A.1); it contains $\underline{\delta e}$, which is also featured by the term following the basic term in (A.1). Before proceeding further, we give $\underline{\delta e}$ a form more convenient than (A.2). In utilizing (9) in (A.2), we have

$$\underline{\delta e} = (\rho^c)^{-1}[\underline{\delta X}_{12} - (\underline{\delta X}_{12}\cdot\underline{e}^c)\underline{e}^c]\ .$$

The expression inside the brackets is that referenced in the preceding paragraph with regard to the decomposition of a vector. It thus follows immediately that

$$\underline{\delta e} = (\rho^c)^{-1}\,\underline{\delta X}_{12,n}\ . \qquad (A.4)$$

This relation could have been transcribed directly from (24).

Next, we express the second term on the right-hand side of (A.1). We use (A.4) as well as the identity (28) and the three lines following it:

$$\underline{\ddot{X}}^c_{12}\cdot\underline{\delta e} = (\rho^c)^{-1}\,\underline{\ddot{X}}^c_{12}\cdot\underline{\delta X}_{12,n} = (\rho^c)^{-1}\,\underline{\ddot{X}}^c_{12,n}\cdot\underline{\delta X}_{12}\ .$$

Thus, the desired term is

$$\underline{c_2 = (\rho^c)^{-1}\,\underline{\ddot{X}}^c_{12,n}\cdot\underline{\delta X}_{12}}\ . \qquad (A.5)$$

To express the third term on the right-hand side of (A.1), we use equation (9'); the notion of a vector decomposition utilized already twice above; and equation (28'):

$$|\dot{\underline{X}}^c_{12}|^2 - (\dot{\rho}^c)^2 = \dot{\underline{X}}^c_{12}\cdot[\dot{\underline{X}}^c_{12} - (\dot{\underline{X}}^c_{12}\cdot\underline{e}^c)\underline{e}^c]$$

$$= \dot{\underline{X}}^c_{12}\cdot\dot{\underline{X}}^c_{12,n} = |\dot{\underline{X}}^c_{12,n}|^2 .$$

Thus, the desired term is

$$\underline{c_3 = -(\rho^c)^{-2}\ |\dot{\underline{X}}^c_{12,n}|^2\ \underline{e}^c\cdot\underline{\delta X}_{12}} . \qquad (A.6)$$

The remaining term (besides the basic term) on the right-hand side of (A.1) is that mentioned earlier, namely the last term inside the second brackets. To give it a more concise form avoiding the explicit presence of $\underline{\delta e}$, we express the latter from (A.4), and use the identity (28) and the three lines following it:

$$\dot{\underline{X}}^c_{12}\cdot\underline{\delta e} = (\rho^c)^{-1}\ \dot{\underline{X}}^c_{12}\cdot\underline{\delta X}_{12,n} = (\rho^c)^{-1}\ \dot{\underline{X}}^c_{12,n}\cdot\underline{\delta X}_{12} .$$

Thus, the desired term is

$$\underline{c_4 = -2(\rho^c)^{-2}\ \dot{\rho}^c\ \dot{\underline{X}}^c_{12,n}\cdot\underline{\delta X}_{12}} . \qquad (A.7)$$

We are now in a position to write (A.1,2) in a convenient form as follows:

$$\underline{\delta\rho = \underline{\delta X}_{12}\cdot\underline{e}^c + c_1 + c_2 + c_3 + c_4} . \qquad (A.8)$$

where the terms c_1, c_2, c_3, and c_4 are given respectively by (A.3,5,6,7). This is precisely the final formulation (29), (30a-d). Although the formal algebraic derivation resulting in (A.1,2) is short and straightforward by comparison with the geometric considerations in Chapters 2-5, the system (A.1,2) is seen to be far from final. Its transformation, carried out with the aid of relations from Chapters 3 and 5, is lengthier than the derivation of the system (A.1,2) itself.

The final formulation above has two basic features. The first feature is that every term on the right-hand side of (A.8) has an easy geometrical interpretation, and is quite concise and tractable. The term c_1 depends on $\underline{\delta\dot{X}}_{12}$, whereas c_2, c_3, and c_4 all depend on $\underline{\delta X}_{12}$ and can be grouped together if convenient. And the second feature is that the term c_1 has been shown in Chapter 6 to be very significant and in general much greater than the combined term $(c_2+c_3+c_4)$, which, in many applications, may be negligible. Thus, a sufficiently accurate yet easily manageable formulation appears to be

$$\underline{\delta\dot{\rho} = \underline{e}^c\cdot\underline{\delta\dot{X}}_{12} + 2(\rho^c)^{-1}\ \dot{\underline{X}}^c_{12,n}\cdot\underline{\delta X}_{12}} . \qquad (A.9)$$

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